From Random Matrices to Quasiperiodic Jacobi Matrices via Orthogonal Polynomials

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Abstract

We present an informal review of results on asymptotics of orthogonal polynomials, stressing their spectral aspects and similarity in two cases considered. They are polynomials orthonormal on a finite union of disjoint intervals with respect to the Szegö weight and polynomials orthonormal on $\mathbb{R}$ with respect to varying weights and having the same union of intervals as the set of oscillations of asymptotics. In both cases we construct double infinite Jacobi matrices with generically quasiperiodic coefficients and show that each of them is an isospectral deformation of another. Related results on asymptotic eigenvalue distribution of a class of random matrices of large size are also shortly discussed.

1 Introduction

The goal of the paper is to discuss a link between asymptotics of a class of orthogonal polynomials, in particular polynomials with respect to varying weights (see e.g. [39]), and the Jacobi matrices with quasi-periodic coefficients, seen mostly as a particular case of ergodic finite-difference operators. The theory of this class of operators owes a lot to B. Simon, starting from an early book [13] till just appeared impressive [36]. The link became clear while the author was reflecting on applications of the asymptotic formulas, found in the remarkable paper by P. Deift et al [15], to certain problems on the eigenvalue distribution of a class of random matrices, known as unitary invariant matrix models. This is why we would like to begin from a discussion of random matrices, despite that the link can be described without a recourse to random matrices.

Consider $n \times n$ Hermitian random matrices

$$M_n = \{M_{jk} \in \mathbb{C},\ M_{kj} = \overline{M_{jk}}\}_{j,k=1}^{n}, \tag{1.1}$$

whose probability law is

$$P(dM_n) = Z_n^{-1} \exp\{-n\mathrm{Tr}V(M_n)\}dM_n. \tag{1.2}$$

Here Z_n is the normalization constant, $V : \mathbb{R} \to \mathbb{R}_+$ is a continuous, bounded below and growing at infinity function (think about a polynomial of an even degree, positive at infinity, see also (2.11)), and

$$dM_n = \prod_{j=1}^{n} dM_{jj} \prod_{1\le j<k\le n} d\Re M_{jk} d\Im M_{jk}. \tag{1.3}$$

This class of random matrices arises in a number of fields of mathematics and physics (see e.g. reviews [16, 18, 20, 25, 26, 27] and references therein). A considerable amount of corresponding problems can be described in terms of the Normalized Counting Measure of eigenvalues (NCM), defined as the relative to n number of eigenvalues of M_n, falling into a given set $\Delta \subset \mathbb{R}$:

$$N_n(\Delta) = \sharp\{\lambda_l^{(n)} \in \Delta,\ l = 1, ..., n\}/n, \tag{1.4}$$

where

$$\{\lambda_l^{(n)}\}_{l=1}^n \tag{1.5}$$

are eigenvalues of M_n.

We note that similar measures arise in spectral theory of ergodic operators, and B. Simon did a lot of excellent work on the measures, whose limit as $n \to \infty$ is known there as the Integrated Density of States.

It will be convenient to consider a bit more general object, known as a linear eigenvalue statistics and defined via a test function $\varphi : \mathbb{R} \to \mathbb{R}$ and eigenvalues (1.5):

$$N_n[\varphi] = n^{-1} \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}). \tag{1.6}$$

We obtain (1.4) by setting $\varphi = \chi_\Delta$, where χ_Δ is the indicator of Δ.

Here are three basic quantities of the Random Matrix Theory, related to eigenvalue statistics and widely studied, especially as $n \to \infty$.

(i) Expectation of $N_n[\varphi]$ with respect to (1.2) – (1.3):

$$\overline{N}_n[\varphi] = \mathbf{E}\{N_n[\varphi]\}. \tag{1.7}$$

(ii) Covariance of N_n for two test functions $\varphi_{1.2}$:

$$\begin{aligned} &\mathbf{Cov}\{N_n[\varphi_1], N_n[\varphi_2]\} \\ &\quad = \mathbf{E}\{N_n[\varphi_1]N_n[\varphi_2]\} - \mathbf{E}\{N_n[\varphi_1]\}\mathbf{E}\{N_n[\varphi_2]\}. \end{aligned} \tag{1.8}$$

(iii) Gap probability

$$E_n(\Delta) = \mathbf{P}\{N_n(\Delta) = 0\}. \tag{1.9}$$

It is a remarkable observation by Gaudin, Mehta, and Dyson of the early 60th (see e.g. [25]) that the above quantities can be expressed via the orthonormal polynomials $\{p_l^{(n)}\}_{l\geq 0}$ with respect to the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{1.10}$$

$$\int w_n(\lambda) p_l^{(n)} p_m^{(n)} d\lambda = \delta_{lm},\ l, m = 0, 1, ... \tag{1.11}$$

Here and below integrals without limits denote integrals over $\mathbb{R}$. We will call V potential. Denote

$$\psi_l^{(n)}(\lambda) = w_n^{1/2}(\lambda) p_l^{(n)}(\lambda), \tag{1.12}$$

and

$$K_n(\lambda,\mu)=\sum_{l=0}^{n-1}\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu). \tag{1.13}$$

K_n is called the reproducing kernel of the orthonormal system $\{\psi_l^{(n)}(\lambda)\}_{l\geq 0}$, and $(K_n(\lambda,\lambda))^{-1}$ is known in the approximation theory as the Christoffel function [3, 34, 39].

To distinguish these polynomials from the traditional ones for which the weight does not contain the large parameter n, the polynomials (1.10) – (1.12) are called the orthogonal polynomials with respect to varying weights (see e.g. [39]).

We have for (1.7) – (1.9):

$$\overline{N}_n[\varphi]=\int\varphi(\lambda)\rho_n(\lambda)d\lambda,\ \rho_n(\lambda)=n^{-1}K_n(\lambda,\lambda), \tag{1.14}$$

see [25],

$$\mathbf{Cov}\{N_n[\varphi_1],N_n[\varphi_2]\}=\frac{1}{2n^2}\int\int\frac{\Delta\varphi_1}{\Delta\lambda}\frac{\Delta\varphi_2}{\Delta\lambda}K_n^2(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2, \tag{1.15}$$

where

$$\frac{\Delta\varphi}{\Delta\lambda}=\frac{\varphi(\lambda_1)-\varphi(\lambda_2)}{\lambda_1-\lambda_2}, \tag{1.16}$$

see [30], and

$$E_n(\Delta)=\det(1-K_n(\Delta)), \tag{1.17}$$

where $K_n(\Delta)$ is the integral operator

$$(K_n(\Delta)f)(\lambda)=\int_\Delta K_n(\lambda,\mu)f(\mu)d\mu,\ \lambda\in\Delta, \tag{1.18}$$

see [25].

We will need two more basic facts on orthonormal polynomials. The first is the Jacobi matrix $J^{(n)}$, associated with the polynomials via the r.h.s. of the three term recurrence relation:

$$\lambda p_l^{(n)}(\lambda)=r_l^{(n)}p_{l+1}^{(n)}(\lambda)+s_l^{(n)}p_l^{(n)}(\lambda)+r_{l-1}^{(n)}p_{l-1}^{(n)}(\lambda),\ r_{-1}=0, \tag{1.19}$$

i.e.,

$$J^{(n)}=\{J_{lm}^{(n)}\}_{l,m=0}^{\infty},\quad J_{lm}^{(n)}=r_l^{(n)}\delta_{l+1,m}+s_l^{(n)}\delta_{l,m}+r_{l-1}^{(n)}\delta_{l-1,m}. \tag{1.20}$$

The second is the Christoffel-Darboux formula

$$K_n(\lambda,\mu)=r_{n-1}^{(n)}\frac{\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu)-\psi_{n-1}^{(n)}(\lambda)\psi_n^{(n)}(\mu)}{\lambda-\mu}. \tag{1.21}$$

Formulas (1.14) – (1.18) and (1.21) show that the asymptotic form of (1.7) – (1.9) as $n\to\infty$ is determined by that of $\psi_{n-1}^{(n)}$ and $\psi_n^{(n)}$. These were found by Deift et al [15] in the case, where V is real analytic (see also [8, 9]). In the same paper the limit of (1.17) was found in the so-called local asymptotic regime (see e.g. [27] for its definition, and [30] for another derivation of this result).

The author of this paper has applied the results of [15] to find an asymptotic form of the covariance (1.8), and to prove an analog of the central limit theorem for linear eigenvalue statistics (1.6). It turned out that this requires a bit more information on the asymptotic formulas of [15], and leads to certain objects, related to spectral theory of

quasiperiodic Jacobi matrices. This is discussed in the paper. Asymptotics of covariance and an analog of the central limit theorem for linear statistics of eigenvalues of random matrices (1.2) – (1.3) will be published elsewhere [28].

The paper is organized as follows. In the next section asymptotic formulas for the "ordinary" polynomials orthogonal with respect to weights whose support is a union of $q \geq 1$ disjoint intervals are shortly discussed following papers [2, 7, 33, 40]. We present then asymptotics found in [15] for polynomials, orthogonal with respect to varying weights for the case, where their oscillatory part is the same union of q intervals. In Section 3 we introduce quasiperiodic Jacobi matrices, associated with the both asymptotics and discuss links between the matrices. We argue that they are so-called "finite-band" Jacobi matrices, widely known in the theory of integrable systems, that they are related by an isospectral deformation, and consider a particular case of polynomial potentials in (1.10), where corresponding Jacobi matrices are periodic. In Section 4 we present a collection of facts on asymptotic eigenvalue distributions of random matrices, that can be written in the terms of the above Jacobi matrices. In Appendix we give a direct proof of the isospectrality of the Jacobi matrices related to asymptotics of both classes of orthogonal polynomials.

2 Asymptotics of orthogonal polynomials

2.1 Ordinary orthogonal polynomials

Consider first the case, where the weight w does not depend on n. In this case all the quantities, related to orthonormal polynomials, do not depend on the super-index (n). We will denote the polynomials and related quantities by the same symbols as in (1.11) – (1.13) and (1.19) – (1.21) but without the super-index (n). Assume that the support σ of the weight is a finite union of disjoint finite intervals:

$$\sigma = \bigcup_{l=1}^{q} [a_l, b_l], \ -\infty < a_1 < b_1 < ... < a_q < b_q < \infty. \tag{2.1}$$

Denote $\mathcal{M}_1(\sigma)$ the set of non-negative unit measures on σ and consider the quadratic functional

$$\mathcal{E}_\sigma[m] = -\int_{\sigma\times\sigma} \log|\lambda - \mu| m(d\lambda) m(d\mu), \ m \in \mathcal{M}_1(\sigma). \tag{2.2}$$

The functional possesses a unique minimizer ν (the equilibrium measure for σ):

$$\min_{m\in\mathcal{M}_1(\sigma)} \mathcal{E}_\sigma[m] = \mathcal{E}_\sigma[\nu].$$

This is a standard variational problem of potential theory, that admits a simple electrostatic interpretation in which m is a distribution of positive charges on a conductor σ and ν is the equilibrium distribution of charges.

It is known (see e.g. [34]) that the problem is equivalent to the relations

$$-2\int_\sigma \log|\lambda - \mu|\nu(d\mu) = -l_\sigma, \ \lambda \in \sigma, \tag{2.3}$$

$$-2\int_\sigma \log|\lambda - \mu|\nu(d\mu) \geq -l_\sigma, \ \lambda \in \mathbb{R} \setminus \sigma, \tag{2.4}$$

that are the Euler-Lagrange equations for (2.2). The quantity $-l_\sigma/2$ is known as the Robin constant, and $e^{l_\sigma/2}$ is the logarithmic capacity of σ.

Set

$$\nu(\lambda) = \nu((\lambda, \infty)), \tag{2.5}$$

and

$$\alpha = \{\alpha_l\}_{l=1}^{q-1}, \ \alpha_l = \nu(a_{l+1}). \tag{2.6}$$

With this notation the asymptotic formulas of [7], Theorem 1, (see also [2, 5, 33, 40]) for analogs of orthonormalized functions (1.12), corresponding to a n-independent weights w, can be written as follows.

Assume that the weight satisfies the Szegö condition

$$\log w \in L^1(\sigma, \nu). \tag{2.7}$$

Then there exist the functions $\mathcal{D}_\sigma : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}_+$, and $\mathcal{G}_\sigma : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}$ such that if λ belongs to the interior of σ, we have

$$\begin{aligned} \psi_n(\lambda) &= (2\mathcal{D}_\sigma(\lambda, n\alpha))^{1/2} \\ &\times \cos\Big(\pi n \nu(\lambda) + \mathcal{G}_\sigma(\lambda, n\alpha)\Big) + o(1), \quad n \to \infty, \end{aligned} \tag{2.8}$$

where the remainder vanishes in the $L^2(\sigma)$-norm, and

$$n\alpha = (n\alpha_1, ...n\alpha_{q-1}). \tag{2.9}$$

Besides, there exist functions $\mathcal{R}_\sigma : \mathbb{T}^{q-1} \to \mathbb{R}_+$, and $\mathcal{S}_\sigma : \mathbb{T}^{q-1} \to \mathbb{R}$, such that the coefficients $\{r_l, s_l\}_{l\geq 0}$ of the corresponding Jacobi matrix J_σ (1.20), that does not depend on the super-index n in this case, have the following asymptotic form

$$r_n = \mathcal{R}_\sigma(n\alpha) + o(1), \ s_n = \mathcal{S}_\sigma(n\alpha) + o(1), \ n \to \infty, \tag{2.10}$$

Note that to find (2.10) one needs weaker asymptotics of $p_n(\lambda)$, those for λ outside σ.

Functions $\mathcal{D}_\sigma, \mathcal{G}_\sigma, \mathcal{R}_\sigma$, and $\mathcal{S}_\sigma$ can be expressed via the $(q-1)$ - dimensional Riemann theta-function (see e.g. formula (A.8) below), associated with the two-sheeted Riemann surface. The surface is obtained by gluing together two copies of the complex plane slit along the gaps $(b_1, a_2), ..., (b_{q-1}, a_q), (b_q, a_1)$ of the support of the measure ν, the last gap goes through the infinity [7]. For another form of $\mathcal{D}_\sigma, \mathcal{G}_\sigma, \mathcal{R}_\sigma$, and $\mathcal{S}_\sigma$ see [35, 33].

The case $q = 1$ in (2.1) of polynomials orthogonal on a single interval dates back to S. Bernstein, Szegö and Akhiezer [37].

The components of the vector $\alpha = \{\alpha_l\}_{l=1}^{q-1}$ are rationally independent generically in σ, thus the sequences $\{\mathcal{D}_\sigma(\lambda, n\alpha)\}_{n\in\mathbb{Z}}$, and $\{\mathcal{G}_\sigma(\lambda, n\alpha)\}_{n\in\mathbb{Z}}$ for any fixed λ and the sequences $\{\mathcal{R}_\sigma(n\alpha)\}_{n\in\mathbb{Z}}$, and $\{\mathcal{S}_\sigma(n\alpha)\}_{n\in\mathbb{Z}}$ are quasiperiodic in n (see [7, 17, 22, 35]). As an early precursor of this fact we mention a result by Akhiezer [1], according to which if σ consists of two intervals, then a certain characteristic of corresponding extremal polynomials of degree n can be expressed via the Jacobi elliptic functions as $n \to \infty$. As a result the characteristic does not converge as $n \to \infty$ but has a set of limit points that fill a specific interval generically in the intervals lengths in (2.1).

2.2 Orthogonal polynomials with respect to varying weights

Let $V : \mathbb{R} \to \mathbb{R}_+$ be real analytic and such that

$$\lim_{|\lambda|\to\infty} V(\lambda)/\log(\lambda^2+1) = \infty. \tag{2.11}$$

Consider orthonormal polynomials (1.10) – (1.12). To describe their asymptotics we introduce the functional (cf (2.2)):

$$\mathcal{E}_V[m] = -\int\int \log|\lambda-\mu| m(d\lambda)m(d\mu) + \int V(\lambda)m(d\lambda), \tag{2.12}$$

defined on the set $\mathcal{M}_1(\mathbb{R})$ of non-negative unit measures on $\mathbb{R}$.

The functional (2.12) possesses a unique minimizer N

$$\min_{m\in\mathcal{M}_1(\mathbb{R})} \mathcal{E}_V[m] = \mathcal{E}_V[N]. \tag{2.13}$$

The variational problem, defined by (2.12), goes back to Gauss and is called the minimum energy problem in the external field V (see recent book [34] for a rather complete account of results and references concerning the problem). The unit measure N minimizing (2.12) is called the equilibrium measure in the external field V because of its evident electrostatic interpretation as the equilibrium distribution of linear charges on the ideal conductor occupying the axis $\mathbb{R}$ and confined by the external electric field of potential V. We stress that the corresponding variational problem determines both the (compact) support σ of the measure and the form of the measure.

The problem is equivalent to the relations [34]

$$\Phi(\lambda) = -l_V,\ \lambda \in \sigma, \tag{2.14}$$

$$\Phi(\lambda) \geq -l_V,\ \lambda \in \mathbb{R}\backslash\sigma, \tag{2.15}$$

where

$$\Phi(\lambda) = V(\lambda) - 2\int_\sigma \log|\lambda-\mu| N(d\mu). \tag{2.16}$$

This should be compared with the variational problem (2.2) of potential theory, where the external field is absent but the support σ is given. This problem can be viewed as a particular case of (2.12), corresponding to a sequence of potentials approaching $V = \chi_\sigma^{-1} - 1$, where χ_σ is the indicator of σ.

The minimum energy problem in the external field arises in various domains of analysis and its applications [14, 34, 39]. We will use here a link with Random Matrix Theory. It was argued by Wigner in the 50th (see [25] for references and discussions), and is shown in [10, 19] that the measure $\overline{N}_n$ of (1.7) converges weakly as $n \to \infty$ to the unique minimizer N in (2.13). Moreover, the random measure (1.4) converges weakly to N with probability 1 as $n \to \infty$.

Assume that V is such that the support of N is a union of q disjoint intervals as in (2.1). Introduce the non-increasing function (cf (2.5))

$$N(\lambda) = N((\lambda,\infty)), \tag{2.17}$$

and the $(q-1)$ - dimensional vector (cf (2.6))

$$\beta = \{\beta_l\}_{l=1}^{q-1},\ \beta_l = N(a_{l+1}). \tag{2.18}$$

With this notation the asymptotics for orthogonal polynomials with varying weight found in [15], Theorem 1.1, can be written as follows. There exist continuous functions $\mathcal{D}_V : \sigma \times T^{q-1} \to \mathbb{R}_+$, and $\mathcal{G}_V : \sigma \times T^{q-1} \to \mathbb{R}$, and $0 < \tau \le 1$ such that if λ belongs to the interior of the support σ (2.1) of N, we have (cf (2.8)):

$$\begin{aligned} \psi_n^{(n)}(\lambda) &= (2\mathcal{D}_V(\lambda, n\beta))^{1/2} \\ &\times \cos\Big(\pi n N(\lambda) + \mathcal{G}_V(\lambda, n\beta)\Big) + O(n^{-\tau}),\ n \to \infty, \end{aligned} \tag{2.19}$$

where $\psi_l^{(n)}(\lambda)$ is defined in (1.12), and $n\beta = (n\beta_1, ..., n\beta_{q-1})$. If λ belongs to the exterior of σ, then $\psi_n^{(n)}$ decays exponentially in n as $n \to \infty$.

Similar asymptotic formulas are valid for coefficients of the Jacobi matrix $J^{(n)}$ of (1.20). Namely, according to [15], formula (1.64), there exist continuous functions $\mathcal{R}_V : \mathbb{T}^{q-1} \to \mathbb{R}_+$ and $\mathcal{S}_V : \mathbb{T}^{q-1} \to \mathbb{R}$ such that we have (cf (2.10))

$$r_n^{(n)} = \mathcal{R}_V(n\beta) + O(n^{-\tau}),\ s_n^{(n)} = \mathcal{S}_V(n\beta) + O(n^{-\tau}),\ n \to \infty. \tag{2.20}$$

It will be argued below that the functions $\mathcal{D}_V, \mathcal{G}_V, \mathcal{R}_V$, and $\mathcal{S}_V$ differ from the functions $\mathcal{D}_\sigma, \mathcal{G}_\sigma, \mathcal{R}_\sigma$ and $\mathcal{S}_\sigma$ of formulas (2.9) and (2.10) of the previous subsection only by a shift in the argument. Hence the main difference in asymptotic formulas of the previous and this subsection is that in the former the "rotation number" and the frequencies are determined by the measure ν (see (2.5) – (2.6)), minimizing the functional (2.2), while in the latter these quantities (see (2.17) – (2.18)) are determined by the measure N, minimizing the functional (2.12).

3 Quasiperiodic Jacobi matrices

3.1 Ordinary orthogonal polynomials

Consider orthonormal polynomials with respect to the weight, whose support is a union of q disjoint intervals (2.1). Denote J_σ the semi-infinite Jacobi matrix, associated with the polynomials and let $\{r_l, s_l\}_{l\ge 0}$ be the non-zero coefficients of J_σ (see (1.19) – (1.20), in which the super-index (n) is omitted). Introduce the double-infinite Jacobi matrix $J_{\sigma,n}$, setting

$$r_{k,n} = \begin{cases} r_{k+n}, & k \ge -n, \\ 0, & k < -n, \end{cases} \qquad s_{l,n} = \begin{cases} s_{k+n}, & k \ge -n, \\ 0, & k < -n. \end{cases} \tag{3.1}$$

We will denote by the same symbol $J_{\sigma,n}$ the selfadjoint operator in $l^2(\mathbb{Z})$, defined by the matrix.

Now the asymptotics (2.10) allow us to define a family of "limiting" double-infinite Jacobi matrices and corresponding selfadjoint operators in $l^2(\mathbb{Z})$. Assume for the sake of definiteness that the components of the $(q-1)$-dimensional vector α of (2.6) are rationally independent. Then for any $x = (x_1, ..., x_{q-1}) \in \mathbb{T}^{q-1}$ there exists a subsequence $\{n_i(x)\}_{i\ge 1}$, such that

$$\lim_{i\to\infty} \{n_i(x)\alpha_l\} = x_l,\ l = 1, ..., q-1, \tag{3.2}$$

where $\{t\}$ denotes the fractional part of $t \in \mathbb{R}$. This and (2.10) imply that for any $k \in \mathbb{Z}$ we have

$$\lim_{i\to\infty} r_{n_i(x)+k} = \mathcal{R}_\sigma(k\alpha + x), \ \lim_{i\to\infty} s_{n_i(x)+k} = \mathcal{S}_\sigma(k\alpha + x). \tag{3.3}$$

In other words the sequence $\{J_{\sigma,n_i(x)}\}_{i\geq 1}$ of selfadjoint operators, defined in $l^2(\mathbb{Z})$ by the double infinite Jacobi matrices with coefficients (3.1), converges strongly to the operator in $l^2(\mathbb{Z})$, defined by the double - infinite Jacobi matrix $J_\sigma(x)$ with coefficients

$$\mathcal{R}_\sigma(k\alpha + x), \quad \mathcal{S}_\sigma(k\alpha + x), \ k \in \mathbb{Z}. \tag{3.4}$$

The matrices $J_\sigma(x)$ arise in spectral theory and integrable systems [17, 22, 38] and is known there as finite band Jacobi matrices.

Write the three-term recursion relation for $J_{\sigma,n}$:

$$r_{n+k}\psi_{n+k+1} + s_{n+k}\psi_{n+k} + r_{n+k-1}\psi_{n+k-1} = \lambda\psi_{n+k}, \ k \geq -n, \ \lambda \in \sigma.$$

Setting here $n = n_i(x)$, using asymptotics (2.8), and taking into account that in the obtained asymptotic equality the coefficients in front of "fast oscillating" expressions $\cos(\pi n_j(x)\nu(\lambda))$ and $\sin(\pi n_j(x)\nu(\lambda))$ should be both zero at the limit $i \to \infty$, we find that for any λ, belonging to the interior of σ, the sequences

$$\{(\mathcal{D}_\sigma(\lambda, k\alpha + x))^{1/2} \cos(\pi\nu(\lambda)k + \mathcal{G}_\sigma(k\alpha + x))\}_{k\in\mathbb{Z}}, \tag{3.5}$$

and

$$\{(\mathcal{D}_\sigma(\lambda, k\alpha + x))^{1/2} \sin(\pi\nu(\lambda)k + \mathcal{G}_\sigma(k\alpha + x))\}_{k\in\mathbb{Z}} \tag{3.6}$$

satisfy the limiting three term recurrence relations, defined by the coefficients (3.4). In other words, sequences (3.5) and (3.6) are generalized eigenfunctions of $J_\sigma(x)$ for every λ, belonging to the interior of σ.

Note now that by general principles [3] the resolution of identity $\mathcal{E}_{J_\sigma}$ of the initial Jacobi matrix J_σ is

$$(\mathcal{E}_{J_\sigma}(d\lambda))_{jk} = \psi_j(\lambda)\psi_k(\lambda)d\lambda, \ j, k \geq 0, \tag{3.7}$$

in particular

$$\int_\sigma (\mathcal{E}_{J_\sigma}(d\lambda))_{jk} = \delta_{jk}. \tag{3.8}$$

Hence, the resolution of identity $\mathcal{E}_{J_{\sigma,n}}$ of $J_{\sigma,n}$, defined by (3.1), is

$$(\mathcal{E}_{J_{\sigma,n}}(d\lambda))_{jk} = \begin{cases} \psi_{n+j}(\lambda)\psi_{n+k}(\lambda)d\lambda, & j, k \geq -n \\ 0, & \text{otherwise.} \end{cases}$$

This and asymptotics (2.8) yield for the weak limit of the above projection-valued measure, the resolution of identity $\mathcal{E}_{J_\sigma(x)}$ of $J_\sigma(x)$:

$$\begin{aligned}(\mathcal{E}_{J_\sigma(x)}(d\lambda))_{jk} &= \chi_\sigma(\lambda)(\mathcal{D}_\sigma(\lambda, j\alpha + x)\mathcal{D}_\sigma(\lambda, k\alpha + x))^{1/2} \\ &\times \cos\Big(\pi\nu(\lambda)(j - k) + \mathcal{G}_\sigma(j\alpha + x) - \mathcal{G}_\sigma(k\alpha + x)\Big)d\lambda, \ j, k \in \mathbb{Z},\end{aligned} \tag{3.9}$$

where χ_σ is the indicator of σ. Denoting $\varphi_j(x) = \pi\nu(\lambda)j + \mathcal{G}(j\alpha + x), \ j \in \mathbb{Z}$, we can write the cosine above as $\cos\varphi_j(x)\cos\varphi_k(x) + \sin\varphi_j(x)\sin\varphi_k(x)$. This shows that the r.h.s. of (3.9) is the linear combination of (3.5) – (3.6). Besides, the equality

$$\int_\sigma (\mathcal{E}_{J_\sigma(x)}(d\lambda))_{jk} = \delta_{jk}, \ j, k \in \mathbb{Z}$$

that can also be proved by the limiting transition $n_i(x) \to \infty$ in (3.8), implies that the union of the sequences (3.5) – (3.6) for all λ of the interior of σ forms a complete system in $l^2(\mathbb{Z})$.

Introducing

$$\Psi_j(\lambda, x) = e^{i\pi\nu(\lambda)j} u_j(\lambda, x), \tag{3.10}$$

where

$$u_j(\lambda, x) = \mathcal{U}(\lambda, j\alpha + x),\ \mathcal{U}(\lambda, x) = \mathcal{D}_\sigma^{1/2}(\lambda, x) e^{i\mathcal{G}_\sigma(\lambda,x)}, \tag{3.11}$$

we conclude from the above that the union of sequences

$$\{\Psi_j(\lambda, x)\}_{j\in\mathbb{Z}},\ \{\overline{\Psi_j(\lambda, x)}\}_{j\in\mathbb{Z}} \tag{3.12}$$

for all λ of the interior of σ also forms a complete system of generalized eigenfunctions of the "limiting" selfadjoint operator $J_\sigma(x)$, acting in $l^2(\mathbb{Z})$. The system is known in spectral theory as the quasi-Bloch generalized eigenfunctions, because in the case of periodic coefficients (see e.g. (3.43)) they are well known Floquet-Bloch solutions of corresponding finite-difference equation. In this context $\nu(\lambda)$ is called the quasi-momentum as the function of spectral parameter.

Recall now that if $A = \{A(x)\}_{x\in\mathbb{T}^{q-1}}$ is a selfadjoint quasiperiodic operator in $l^2(\mathbb{Z})$, then its Integrated Density of States can be defined as

$$k_A(\lambda) = \int_{\mathbb{T}^{q-1}} (\mathcal{E}_{A(x)}((\lambda, \infty))_{00}\, dx, \tag{3.13}$$

where $\mathcal{E}_{A(x)}$ is the resolution of identity of $A(x)$ (see [13, 29] for this and a more general case of ergodic operators).

By using the above definition and (3.7), we find

$$k_{J_\sigma(\cdot)}(d\lambda) = \left(\int_{\mathbb{T}^{q-1}} \mathcal{D}_\sigma(\lambda, x) dx\right) d\lambda, \tag{3.14}$$

where $k_{J_\sigma(\cdot)}(d\lambda)$ is the measure, corresponding to the non-increasing function $k_{J_\sigma(\cdot)}(\lambda)$ in (3.13).

Another definition of the Integrated Density of States is as follows. Consider the restriction $A_n(x)$ of $A(x)$ to a finite interval $[1, n]$, imposing certain selfadjoint boundary conditions at the endpoints of the interval. The spectrum of $A_n(x)$ is a finite set and we can introduce its Normalizes Counting Measure of eigenvalues k_n as the divided by n number of eigenvalues of $A_n(x)$ in the interval (λ, ∞) (cf (1.4)). It is known (see e.g. [13, 29]) that k_n converges weakly to (3.13) for any $x \in \mathbb{T}^{q-1}$.

In the case of operators $J_\sigma(x)$, possessing the complete family of quasi Bloch generalized eigenfunctions (3.5) – (3.6), it can be shown that $k_{J_\sigma(\cdot)} = \nu$, i.e. that the Integrated Density of States of $J_\sigma(\cdot)$ coincides with its quasi-momentum as the function of the spectral parameter, and we have from (3.14)

$$\begin{aligned} k_{J_\sigma(\cdot)}(d\lambda) &= \nu(d\lambda) \\ &= \left(\int_{\mathbb{T}^{q-1}} \mathcal{D}_\sigma(\lambda, x) dx\right) d\lambda. \end{aligned} \tag{3.15}$$

We found, in particular, a relation between two quantities of asymptotics (2.8).

Another important characteristics of quasi periodic (more generally, ergodic) operators is the Lyapunov exponent $\gamma_A(\lambda)$, defined as the rate of exponential growth of the Cauchy solutions of the corresponding finite difference equation of second order. The Lyapunov exponent and the Integrated Density of States are related by the Thouless formula (see e.g. [29], formula (11.82)). In the case of the quasiperiodic Jacobi matrix $J_\sigma(x)$ the formula is

$$\gamma_{J_\sigma(\cdot)}(\lambda) = -\int_{\mathbb{T}^{q-1}} \mathcal{R}_\sigma(x)dx + \int_\sigma \log|\lambda - \mu| k_{J_\sigma(\cdot)}(d\lambda). \tag{3.16}$$

Since the generalized functions of $J_\sigma(x)$ are bounded and do not decay at infinity (see (3.5) – (3.6) or (3.10) – (3.11)), we have

$$\gamma_{J_\sigma(\cdot)}(\lambda) = 0, \ \lambda \in \sigma.$$

Hence, the l.h.s. of (3.16) is zero if $\lambda \in \sigma$. In view of (3.14) the obtained relation is just the Euler-Lagrange equation (2.3) for the functional (2.2).

3.2 Orthogonal polynomials with respect to varying weights

We will present here constructions, similar to those of the previous subsection but for orthogonal polynomials with respect to varying weights. To this end it is useful to make explicit the amplitude of the potential V in (1.10), i.e. to replace V by $V/g, \ g > 0$. In what follows we will keep V fixed and vary g. Thus orthonormal polynomials (1.11) and related quantities will depend on g. To control this dependence we will use results of papers [12, 21].

Note first that if the potential is real analytic, then the minimizer N of (2.12) possesses a density ρ supported on a finite union of finite intervals σ [10, 14].

According to [15] asymptotics (2.19) – (2.20) are most precise and well behaving if a real analytic potential, satisfying (2.11), is regular (see [15], formulas (1.12) and (1.13)). This condition implies, in particular, that the density of the measure N in (2.13) is strictly positive on the interior of its support σ, and vanishes as a square root at each edge of σ. Furthermore, following [21], we say that g is regular for V if V/g is a regular potential. If g_0 is regular for V, and

$$\sigma_{g_0} = \bigcup_{l=1}^{q} [a_l(g_0), b_l(g_0)]$$

is the support of the equilibrium measure N_{g_0}corresponding to V/g_0, then there exists an open neighborhood of g_0, consisting of regular values g for V and

$$\sigma_g = \bigcup_{l=1}^{q} [a_l(g), b_l(g)] \tag{3.17}$$

with the same number q of intervals. Besides, a_l and b_l are real analytic, a_l is strictly decreasing and b_l is strictly increasing in g.

We will also need the following formula, relating N_g and ν_g, minimizing correspondingly (2.12) with V/g instead V and (2.2) with σ_g instead of σ [12]:

$$N_g = g^{-1} \int_0^g \nu_{g'} dg'. \tag{3.18}$$

The formula was proved in [12] in a fairly general setting, including piece-wise continuous V's. Its particular cases are given in [26, 11], where its spectral and asymptotic meaning made explicit, related to a kind of "adiabatic" regime in g for corresponding Jacobi matrix (1.20) (see also the derivation of formula (4.3) below).

Now we can give an analog of constructions of previous subsection, i.e. the "limiting" Jacobi matrix with quasiperiodic coefficients. We confine ourselves again to the case of rationally independent components of vector β of (2.18), a generic case in g.

Consider the coefficients $r_l^{(n)}$ of the Jacobi matrix (1.20), associated with orthonormal polynomials $\{p_l^{(n)}\}_{l\geq 0}$ with varying weight. Introducing explicitly the dependence of coefficients on g and writing in view of (1.10) with V/g instead of V

$$n\frac{V}{g} = l\frac{V}{gl/n}, \tag{3.19}$$

we get

$$r_l^{(n)}(g) = r_l^{(l)}(gl/n). \tag{3.20}$$

Setting here $l = n_i(x) + k$, where now (cf (3.2))

$$\lim_{i\to\infty}\{n_i(x)\beta_l\} = x_l,\ l = 1, .., q-1, \tag{3.21}$$

$x = \{x_l\}_{l=1}^{q-1}$ is a point of $\mathbb{T}^{q-1}$, and k is an arbitrary fixed integer, we obtain in view of (2.20) and the continuity of $\mathcal{R}_V$ in g and x, and of β in g:

$$\begin{aligned}\lim_{i\to\infty} r_{n_i(x)+k}^{(n_i(x))}(g) &= \lim_{i\to\infty}\mathcal{R}_V\left(\frac{n_i(x)+k}{n_i(x)}g, (n_i(x)+k)\beta\left(\frac{n_i(x)+k}{n_i(x)}g\right)\right) \\ &= \mathcal{R}_V\left(g, k\widetilde{\alpha}(g)+x\right).\end{aligned} \tag{3.22}$$

where

$$\widetilde{\alpha}(g) = (g\beta(g))'. \tag{3.23}$$

Analogous relation is valid for the diagonal entries of $J^{(n)}$:

$$\lim_{i\to\infty} s_{n_i(x)+k}^{(n_i(x))}(g) = \mathcal{S}_V\left(g, k\widetilde{\alpha}(g)+x\right).$$

Now, by using formula (3.18), we find an important relation

$$\widetilde{\alpha}(g) = \alpha(g), \tag{3.24}$$

where $\alpha(g)$ is defined by (2.6) with ν_g instead of ν. We conclude from the above that limiting coefficients are (cf (3.3))

$$\begin{aligned}\lim_{i\to\infty} r_{n_i(x)+k}^{(n_i(x))}(g) &= \mathcal{R}_V\left(g, k\alpha(g)+x\right),\ k\in\mathbb{Z}, \\ \lim_{i\to\infty} s_{n_i(x)+k}^{(n_i(x))}(g) &= \mathcal{S}_V\left(g, k\alpha(g)+x\right),\ k\in\mathbb{Z}.\end{aligned} \tag{3.25}$$

As a result we obtain a quasiperiodic Jacobi matrix $J_{V/g}(x)$, defined by the coefficients (cf (3.4))

$$\mathcal{R}_V\left(g, k\alpha(g)+x\right),\ \mathcal{S}_V\left(g, k\alpha(g)+x\right),\ k\in\mathbb{Z}, \tag{3.26}$$

and having the frequencies $(\alpha_1(g), ..., \alpha_{q-1}(g))$, obtained from (2.5) – (2.6) in which ν is ν_g, hence σ is the support σ_g of N_g. Note that $J_{V/g}(x)$ is the limit in the sense (3.21) in the strong operator topology of $l^2(\mathbb{Z})$ of matrices $J^{(n)}_{V/g,n}$, whose coefficients are defined by formulas (3.1) with $r^{(n)}_{n+k}$ and $s^{(n)}_{n+k}$ instead of r_{n+k} and s_{n+k}.

Applying the same limiting argument to asymptotic formula (2.19), and by using (3.21) and (3.18), we obtain for any fixed $k \in \mathbb{Z}$

$$\begin{aligned}\psi^{(n)}_{n_i(x)+k}(\lambda) &= (2\mathcal{D}_V(\lambda, g, k\alpha(g)+x))^{1/2} \cos\Big(\pi n_i(x) N_g(\lambda) \\ &+ \pi k \nu_g(\lambda) + \mathcal{G}_V(\lambda, g, k\alpha(g)+x)\Big) + o(1),\ n_i(x) \to \infty.\end{aligned} \tag{3.27}$$

By using these formulas, the exponential decay of $\psi^{(n)}_{n+k}(\lambda)$ outside σ_g, and the limit (3.21), we obtain the complete system of generalized eigenfunctions and the resolution of identity $\mathcal{E}_{J_{V/g}(x)}$ of $J_{V/g}(x)$, given by (3.5) – (3.6) (or (3.12)) and (3.7), in which subindex σ is replaced by the subindex V/g. In particular, we have for the diagonal entries of $\mathcal{E}_{J_{V/g}(x)}$:

$$\left(\mathcal{E}_{J_{V/g}(x)}\right)_{kk}(d\lambda) = \chi_{\sigma_g}(\lambda)\mathcal{D}_V(\lambda, g, k\alpha(g)+x),\ k \in \mathbb{Z}, \tag{3.28}$$

where χ_σ is the indicator of σ (cf (3.9) with $j = k$). The limit here is the weak limit of measures. The support in λ of the r.h.s. of this formula is the support σ_g of the equilibrium measure N_g. This implies that the spectrum of the quasiperiodic matrix $J_{V/g}(x)$ is σ_g. Note that the spectrum of the "initial" double infinite matrix $J^{(n)}_n$, defined analogously (3.1) but via $r^{(n)}_{n+k}$ and $s^{(n)}_{n+k}$, is $\mathbb{R}$ for all $n < \infty$.

Besides, arguing as in obtaining (3.14), we find for the Integrated Density of States measure $k_{J_{V/g}(\cdot)}$ of $J_{V/g}(x)$:

$$\begin{aligned}k_{J_{V/g}(\cdot)}(d\lambda) &= \nu_g(d\lambda) \\ &= \left(\int_{\mathbb{T}^{q-1}} \mathcal{D}_V(\lambda, g, x)dx\right) d\lambda.\end{aligned} \tag{3.29}$$

Comparing the first equality of this formula with the first equality of (3.15) in which σ is replaced by σ_g, we conclude that $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ have the same spectrum and the same Integrated Density of States.

The coincidence of spectra of $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ implies (see [17, 22, 38]) that each of them is an isospectral deformation of another, i.e. that the coefficients (3.4) of $J_{\sigma_g}(x)$ differ from the coefficients (3.26) of $J_{V/g}(x)$ just by a shift of their argument. This fact can also be checked directly, by comparing explicit formulas for both sets of coefficients, given [7, 17, 38] and in [15] correspondingly, and by using again the trick with infinitesimal variation of the amplitude of potential (see Appendix).

Here is one more link between two classes of polynomials and spectral theory. It concerns the Lyapunov exponents of $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ and the potential. It can be shown [12] that the Lyapunov exponents of the both matrices coincide and if $\gamma_g(\lambda)$ is their common value, then

$$V(\lambda) = 2\int_0^g \gamma_{g'}(\lambda)dg',\ \lambda \in \sigma_g. \tag{3.30}$$

3.3 Periodic Jacobi matrices

Here we consider a class of polynomial potentials V in (1.10) for which corresponding Jacobi matrices $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ have periodic coefficients. Besides, several quantities related to the matrices and orthogonal polynomials can be found explicitly. We follow [11].

Let v be a polynomial of degree q with real coefficients and with the leading term z^q. Assume that there exists $g > 0$ such that all zeros of the polynomial $v^2 - 4g$ are real and simple and set

$$V(\lambda) = \frac{v^2(\lambda)}{2q}. \tag{3.31}$$

We will show that in this case coefficients of $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ are q-periodic and their spectrum is

$$\sigma_g = \{\lambda : v^2(\lambda) - 4g \le 0\}. \tag{3.32}$$

We show first that the equilibrium measures N_g and ν_g for (2.12) and (2.2) are:

$$N_g(d\lambda) = \rho_g(\lambda)d\lambda,\ \rho_g(\lambda) = \frac{|v'(\lambda)|}{2\pi gq}|v^2(\lambda) - 4g|^{1/2}\chi_{\sigma_g}(\lambda), \tag{3.33}$$

and

$$\nu_g(d\lambda) = d_g(\lambda)d\lambda,\ d_g(\lambda) = \frac{|v'(\lambda)|}{\pi q}|v^2(\lambda) - 4g|^{-1/2}\chi_{\sigma_g}(\lambda). \tag{3.34}$$

Indeed, it is a matter of direct calculations to find that if ρ_g is given by (3.33), then

$$\begin{aligned} w_{V/g}(z) :&= -\int_{\sigma_g} \log(z-\mu)\rho_g(\mu)d\mu = \frac{1}{q}\Big[u(z)\sqrt{u^2(z)-1} \\ &- \log\left(u(z) + \sqrt{u^2(z)-1}\right) - u^2(z)\Big] - (2q)^{-1}\log g/e. \end{aligned} \tag{3.35}$$

where $u = v/2\sqrt{g}$, and we use the branch of logarithm with the cut $(-\infty, 0)$ and the argument π on the upper edge of the cut and the branch of $\sqrt{u^2-1}$, such that $\sqrt{u^2-1} = u + o(1),\ u \to \infty$.

On the other hand

$$\Re w_{V/g}(\lambda + i0) = -\int_{\sigma_g} \log|\lambda - \mu|\rho_g(\mu)d\mu$$

is the logarithmic potential of $N_g(d\lambda) = \rho_g(\lambda)d\lambda$. Now, analyzing the values of Φ of (2.16) in this case

$$\Phi_g(\lambda) = \frac{V(\lambda)}{g} + 2\Re w_{V/g}(\lambda + i0) \tag{3.36}$$

with V from (3.31), we can check directly the validity of (2.14) – (2.15) (with V/g instead of V) with the strict inequality in (2.15) and $l_V = (2q)^{-1}\log g/e$. This proves that N_g of (3.33) is the minimizer of (2.12) with V/g instead of V.

It can also be proved that ν_g of (3.34) is the minimizer of (2.2) with σ_g of (3.32) instead of σ. We can use either (3.18) or the above scheme, computing (cf (3.35))

$$\begin{aligned} w_{\sigma_g}(z) :&= -\int_{\sigma_g} \log(z-\mu)d_g(\mu)d\mu \\ &= -\frac{1}{q}\log\left(u(z) + \sqrt{u^2(z)-1}\right) - \frac{1}{2q}\log g, \end{aligned}$$

and then checking directly (2.3) – (2.4).

We will use, however, another argument to prove (3.33) and (3.34). The argument is based on a representation, important in the inverse problem for periodic operators of second order [24, 23].

Let u be a polynomial of degree q with real coefficients and such that all zeros of u^2-1 are real and simple. Then u can be written in the form

$$u(z) = \cos\theta(z), \tag{3.37}$$

in which $\theta(z)$ is the conformal map of the open upper half-plane $\mathbb{C}_+ = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ onto the domain

$$\begin{aligned} \{\theta \ : \ & q_1\pi < \Re\theta < q_2\pi,\ \Im\theta > 0\} \\ & \setminus \bigcup_{q_1<l<q_2,} \{\theta : \Re\theta = l\pi,\ q_1 < l < q_2,\ 0 < \Im\theta \le h_l\}, \end{aligned} \tag{3.38}$$

Here $q_1 < q_2$ are integers, $q_2 - q_1 = q$, $0 \le h_l \ < \infty$ and $\theta(\infty) = \infty$. In fact, the r.h.s. of (3.37) is a polynomial of degree q if and only if $-\infty < q_1 < q_2 < \infty,\ q_2 - q_1 = q$ [24]. Function $\theta(z)$ is analytic in $\mathbb{C}_+$ and continuous in the closed upper half-plane $\overline{\mathbb{C}_+}$. When $z = \lambda + i0$ varies from $-\infty$ to ∞, the limiting value $\theta(\lambda + i0)$ runs along the boundary (the "comb") of the domain (3.38), so that either $\Re\theta(\lambda + i0)$ varies from $(q_1 + l - 1)\pi$ to $(q_1 + l)\pi$ and $\Im\theta(\lambda + i0) = 0$, if λ varies through the lth "band" $[a_l, b_l]$, $l = 1, ..., q$ of σ_g, or $\Re\theta(\lambda + i0) \equiv 0 \ (\mathrm{mod}\pi)$ and $\Im\theta(\lambda + i0) = \kappa,\ 0 \le \kappa \le h_l$, if λ varies through the lth "gap" (b_l, a_{l+1}) of σ_g.

By using the terminology of mathematical physics we can say that $\theta(z)/\pi q$ is an analytic continuation of the quasimomentum as a function of energy in the extended band scheme.

We set $u = v/2\sqrt{g}$ in (3.37). Then the zeros of $u^2 - 1$ are the band edges $-\infty < a_1 < b_1 < ... < a_q < b_q,\ \theta(b_q) = 0,\ q_1 = -q,\ q_2 = 0$ and $\theta(\lambda + i0)$ varies from $(-q + l - 1)\pi$ to $(-q + l)\pi,\ l = 1, ..., q$ when λ varies from a_l to b_l in the lth band. By using (3.37) and (3.38) we can rewrite (2.14) – (2.16), (3.36) as

$$\Phi(\lambda) = -\frac{1}{q}\log g/e + \left\{ \begin{array}{rl} 0, & \text{if } \lambda \in [a_l, b_l],\ l = 1, ..., q, \\ g(\kappa_l(\lambda)), & \text{if } \lambda \in [b_l, a_{l+1}],\ l = 1, .., q, \end{array} \right. \tag{3.39}$$

where $a_{l+1} = a_1$,

$$g(\kappa) := \frac{2}{q}\left(\frac{\sinh 2\kappa}{2} - \kappa\right) = \frac{4}{q}\int_0^\kappa \sinh^2 t dt > 0,\ \kappa > 0,$$

and $\kappa_l(\lambda)$ varies from 0 and $h_l > 0$, when λ varies through the gap (b_l, a_{l+1}). This yields (2.14) – (2.16), thereby a proof that (3.33) is the density of the equilibrium measure N_g, corresponding to the potential (3.31). Moreover, since the inequality in (2.15) is strict in this case, the corresponding value of g is regular for the potential (3.31).

It follows from (3.35) that

$$N_g(\lambda) = -\pi^{-1}\Im w_{V/g}(\lambda + i0),$$

and then (3.37) implies that

$$N_g(\lambda) = \frac{1}{\pi q}\left(\theta_+(\lambda) - \frac{\sin 2\theta_+(\lambda)}{2}\right), \tag{3.40}$$

and similarly

$$\nu_g(\lambda) = \frac{1}{\pi q}\theta_+(\lambda), \tag{3.41}$$

where $\theta_+(\lambda) = \Re\theta(\lambda + i0)$, and $\theta(z)$ is defined in (3.37). In view of the above properties of this function, we have

$$N_g(a_{l+1}) = \nu_g(a_{l+1}) = \frac{q-l}{q}, \quad l = 1, ..., q-1, \tag{3.42}$$

and then (2.6) and (2.18) imply

$$\alpha_l = \beta_l = \frac{q-l}{q}, \quad 1 = 1, ..., q-1. \tag{3.43}$$

Hence, the coefficients (3.4) of the matrix $J_{\sigma_g}(x)$ and the coefficients (3.26) of the matrix $J_{V/g}(x)$ are q-periodic in this case. Moreover, we need not to consider in this case the whole torus $\mathbb{T}^{q-1}$ as the set of values of x in (3.2) and (3.21), but just the set of vertices of the regular q-polygon. This is similar to a standard procedure of the theory of almost periodic functions, where the corresponding set is the closure of all limiting points of sequences (3.2) or (3.21), hence depends on arithmetic properties of the frequency vector α or β.

Consider simple cases of potentials (3.31). The case $q = 1$ corresponds to $v(\lambda) = -\lambda$ and yields

$$\begin{aligned}
\sigma_g &= [-2\sqrt{g}, 2\sqrt{g}], \\
\rho_g(\lambda) &= \frac{1}{2\pi g}(4g - \lambda^2)^{1/2}\chi_{\sigma_g}(\lambda), \\
d_g(\lambda) &= \frac{1}{\pi}(4g - \lambda^2)^{-1/2}\chi_{\sigma_g}(\lambda).
\end{aligned}$$

The first density corresponds to the well known semicircle law by Wigner for the Gaussian Unitary Ensemble [25]. The role of polynomials $p_l^{(n)}$ play $h_l(\lambda\sqrt{n/2g})(n/2g)^{1/4}$, where $\{h_l\}_{l\geq 0}$ are the orthonormal Hermite polynomials. The second density is the Density of States of the Jacobi matrix with constant coefficients $r_l = \sqrt{g},\ s_l = 0,\ l \in \mathbb{Z}$. The matrix plays here the role of both limiting matrices J_{σ_g} and $J_{V/g}$.

The case $q = 2$ corresponds to $v(\lambda) = \lambda^2 + v_0,\ v_0 < -2\sqrt{g}$, and yields

$$\begin{aligned}
\sigma_g &= [-b(g), -a(g)] \cup [a(g), b(g)], \\
a(g) &= (|v_0| - 2\sqrt{g})^{1/2},\ b(g) = (|v_0| + 2\sqrt{g})^{1/2}, \\
\rho_g(\lambda) &= \frac{|\lambda|}{2\pi g}\left((b^2 - \lambda^2)(\lambda^2 - a^2)\right)^{1/2}\chi_{\sigma_g}(\lambda), \\
d_g(\lambda) &= \frac{|\lambda|}{\pi}\left((b^2 - \lambda^2)(\lambda^2 - a^2)\right)^{-1/2}\chi_{\sigma_g}(\lambda).
\end{aligned}$$

Asymptotics of corresponding orthogonal polynomials were considered in [8]. Matrices J_{σ_g} and $J_{V/g}$ are both of period 2 and their Density of States is given above.

For a general two interval case, where the corresponding matrices are quasiperiodic and their coefficients can be expressed via the Jacobi elliptic functions see [2, 5, 32] (ordinary polynomials) and [9] (polynomials with varying weights).

The fact that in the case of ordinary polynomials the limiting finite band Jacobi matrix is periodic if its spectrum is the inverse image of a polynomial map (see (3.32)) is known (see e.g. [31, 38] and references therein). It is of interest that the same property holds also for polynomials with varying weights and that the corresponding potential (3.31) is also polynomial and can be explicitly related to the map.

We mention one more link of asymptotics of orthonormal polynomials and periodic Jacobi matrices [11] that concerns the Hill discriminant (or the Lyapunov function) of $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ and the polynomial v of (3.31). Recall that the Hill discriminant $\Delta(\lambda)$ is defined as 1/2 of the trace of the monodromy (transfer) matrix of corresponding finite-difference equation of second order with periodic coefficients and plays an important role in spectral theory (see e.g. [23, 36]). It can be shown [11] that both matrices have the same Hill discriminant Δ_g and that

$$\Delta_g = v(\lambda)/2\sqrt{g}.$$

We discussed above the case, where the polynomial v in (3.31) is such that all zeros of $v^2 - 4g$ are real and simple. Admitting non-simple (but still real) zeros, we include the case, where two adjacent bands touch one another or a band is going to appear inside a gap.

4 Eigenvalue distribution of random matrices

Here we discuss briefly certain aspects of eigenvalue distributions of ensembles (1.1) – (1.3), related to the above results, in particular to the matrix $J_{V(x)}$.

4.1 Expectation of linear statistics

According to [10, 19] we have for any bounded and continuous φ

$$\lim_{n\to\infty} \mathbf{E}\{N_n[\varphi]\} = \int_\sigma \varphi(\lambda)N(d\lambda),$$

where N is the minimizer of (2.12). Combining this with (3.18), (3.28), and 3.29), we obtain for the r.h.s. of this formula

$$\int_\sigma \varphi(\lambda)d\lambda \int_0^1 \nu_g(d\lambda)dg = \int_0^1 dg \int_{\mathbb{T}^{q-1}} (\varphi(J_{V/g}(x)))_{00}\, dx.$$

On the other hand we can always write (1.6) as

$$N_n[\varphi] = n^{-1}\mathrm{Tr}\varphi(M_n),$$

and we obtain a kind of "functional correspondence"

$$\lim_{n\to\infty} \mathbf{E}\{n^{-1}\mathrm{Tr}\varphi(M_n)\} = \int_0^1 dg \int_{\mathbb{T}^{q-1}} (\varphi(J_{V/g}(x)))_{00}\, dx, \tag{4.1}$$

reminiscent to that for ergodic operators, see [13], Theorem 9.6, and [29], Theorem 4.4.

Here is a heuristic argument, explaining the above formula. According to (1.14) the l.h.s. of the formula includes the orthonormal functions $\psi_l^{(n)}$ of (1.12) for $l = 0, ..., n-1$.

Indicating explicitly the dependence of these functions on g and using the relation (cf (3.20)):

$$\psi_l^{(n)}(\lambda, g) = \psi_l^{(l)}(\lambda, gl/n), \tag{4.2}$$

we obtain from (2.19) that the leading contribution to ρ_n as $n \to \infty$ is

$$\frac{1}{n}\sum_{l=0}^{n-1} \mathcal{D}_V\left(\lambda, l/n, l\beta(l/n)\right),$$

Assuming that $\mathcal{D}_V(\lambda, g, x)$, and $\beta(g)$ are continuous in g, we can say that the summand in this formula is "slow varying" in l/n and "fast varying" in l. This observation results in the limiting formula for the density $\rho(\lambda)$ of the measure N:

$$\rho(\lambda) = \int_0^1 dg \int_{\mathbb{T}^{q-1}} \mathcal{D}_V(\lambda, g, x)dx. \tag{4.3}$$

Using now (3.29) and (3.18), we obtain (4.1).

4.2 Covariance of linear statistics of eigenvalues

By using (1.21) we write (1.15) as

$$\mathbf{Cov}\{N_n[\varphi_1], N_n[\varphi_2]\} = \frac{1}{n^2}\int\int \frac{\Delta\varphi_1}{\Delta\lambda}\frac{\Delta\varphi_2}{\Delta\lambda}\mathbf{C}_n(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2,$$

where $\Delta\varphi/\Delta\lambda$ is defined in (1.16),

$$\mathbf{C}_n(\lambda_1, \lambda_2) = (r_{n-1}^{(n)})^2\left(e_{n,n}^{(n)}(\lambda_1)e_{n-1,n-1}^{(n)}(\lambda_2) - e_{n,n-1}^{(n)}(\lambda_1)e_{n,n-1}^{(n)}(\lambda_2)\right)$$

and

$$e_{l,m}^{(n)}(\lambda) = \psi_l^{(n)}(\lambda)\psi_m^{(n)}(\lambda)$$

is the density of the resolution of identity $\{\mathcal{E}_{J^{(n)}}(d\lambda)\}_{l,m=0}^{\infty}$ of matrix (1.20) (cf (3.7)). Thus, assuming that $\varphi_{1,2}$ are bounded and of the class C^1 and passing to a subsequence $\{n_i(x)\}_{i\geq 1}$ that satisfies (3.21), we obtain in view of (3.22)

$$\begin{aligned}\lim_{n_i(x)\to\infty} n_i(x)^2\mathbf{Cov}\{N_{n_i(x)}[\varphi_1], N_{n_i(x)}[\varphi_2]\} &= \mathcal{R}_V^2(x-\alpha) \\ \times\int_{\sigma\times\sigma} \frac{\Delta\varphi_1}{\Delta\lambda}\frac{\Delta\varphi_2}{\Delta\lambda}\Big(e_{0,0}(\lambda_1)e_{-1,-1}(\lambda_2) &- e_{0,-1}(\lambda_1)e_{0,-1}(\lambda_2)\Big)d\lambda_1 d\lambda_2,\end{aligned} \tag{4.4}$$

where $e_{j,k}(\lambda),\ j,k \in \mathbb{Z}$ is the density of the (jk)th entry of the resolution of identity $\{\mathcal{E}_{J_V(x)}(d\lambda)\}_{l,m\in\mathbb{Z}}$ of the limiting Jacobi matrix $J_V(x)$, determined by (3.26).

This result seems rather unusual from the point of view of traditional probability concepts. Indeed, the covariance of linear eigenvalue statistics (1.6) is of the order n^{-2} rather than of the order n^{-1} as in the case of independent identically distributed random variables, or in a more complex and close to our context case of the Schrodinger operator with random potential. In the latter case it is a matter of routine spectral theory argument to show that, say for

$$\varphi_z(\lambda) = (\lambda - z)^{-1},\ \Im z \neq 0, \tag{4.5}$$

where

$$N_n[\varphi_z] = n^{-1}\mathrm{Tr}(H_n - z)^{-1}$$

and H_n is the discrete Schrodinger operator on the interval $[1, n]$ with a random i.i.d. potential, then

$$\lim_{n\to\infty} n\mathbf{Cov}\{n^{-1}\mathrm{Tr}(H_n - z_1)^{-1}, n^{-1}\mathrm{Tr}(H_n - z_2)^{-1}\} = C(z_1, z_2),$$

where $C(z_1, z_2)$ is analytic for $\Im z_{1,2} \neq 0$ and is not identical zero if the second moment of the potential exists.

On the other hand, it follows from (4.4) that if M_n is a random $n \times n$ random matrix, given by (1.1) – (1.3), then

$$\lim_{n_i(x)\to\infty} n_i^2(x)\mathbf{Cov}\{n_i(x)^{-1}\mathrm{Tr}(M_n - z_1)^{-1}, n_i(x)^{-1}\mathrm{Tr}(M_n - z_2)^{-1}$$
$$= \mathcal{R}_V^2(x-\alpha)\left(\frac{\Delta G_{0,0}(x)}{\Delta z}\frac{\Delta G_{-1,-1}(x)}{\Delta z} - \left(\frac{\Delta G_{0,-1}(x)}{\Delta z}\right)^2\right),$$

where

$$\frac{\Delta G_{j,k}(x)}{\Delta z} = \frac{1}{z_1 - z_2}\left((J_{V(x)} - z_1)^{-1} - (J_{V(x)} - z_2)^{-1}\right)_{j,k}.$$

Hence the covariance of $N_n[\varphi_z] = n^{-1}\mathrm{Tr}(M_n - z)^{-1}$ is of the order $O(n^{-2})$.

This indicates that the Central Limit Theorem, if any, should be valid not for $n^{1/2}N_n[\varphi_z]$ as in the case of i.i.d. random variables, but for $nN_n[\varphi_z]$, i.e. for the sum

$$\mathcal{N}_n[\varphi_z] = \sum_{l=1}^{n} \varphi(\lambda_l^{(n)})$$

without a n-dependent factor in front.

This was indeed shown in [19] for the single interval case $q = 1$ and for a rather broad class of test functions. However, as we have seen above, the case $q = 1$ is exceptional, since it is only in this case asymptotic formulas (2.19) and (2.20) do not oscillate in n because of the absence of the argument $n\beta$ in corresponding coefficients of the formulas. Hence, for $q \geq 2$ the limiting normal law for $\mathcal{N}_n[\varphi_z]$, if it exists, could be different for subsequences in (3.2) having different limits in $\mathbb{T}^{q-1}$, because its variance depends on $x \in \mathbb{T}^{q-1}$:

$$\lim_{n_i(x)\to\infty} \mathbf{Var}\{\mathcal{N}_{n_i(x)}[\varphi_z]\}$$
$$= \mathcal{R}_V^2(x-\alpha)\Big(g_{0,0}(z,x)g_{-1,-1}(z,x) - |g_{0,-1}(z,x)|^2\Big),$$

where

$$g_{jk}(z,x) = \int_\sigma \frac{(\mathcal{E}_{J_V(x)}(d\lambda))_{j,k}}{|\lambda - z|^2}.$$

However, as is shown in [28], the situation with the Central Limit Theorem for linear eigenvalue statistics of random matrices (1.2) – (1.3) is more subtle. Namely, the above scheme of a family of the Gaussian limiting law with the x-dependent variance for various subsequences in (3.21) proves to be valid in the case, where the matrix $J_V(x)$ is periodic. In a generic case of quasiperiodic $J_V(x)$ the limiting laws of subsequences $\{\mathcal{N}_{n_i(x)}[\varphi]\}_{i\geq 1}$ exist but are not Gaussian.

A Appendix

Here we verify directly that the coefficients of the Jacobi matrices $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ coincide up to a shift in x. We will consider again the generic case, where the frequencies (2.6) and (2.18) are rationally independent, hence x varies over the whole $\mathbb{T}^{q-1}$. Besides, we consider only the off-diagonal entries of $J_{\sigma_g}(x)$ of (3.4) and $J_{V/g}(x)$ of (3.26) (note that the diagonal entries are zero if V is even).

Recall that the coincidence follows also from general results on the inverse problem of spectral analysis for "finite-band" potentials, known as the algebro-geometric approach (see e.g. [17, 22, 38]). Indeed, since the spectra $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ coincide (see Section 3.2), $J_{\sigma_g}(x)$ is a isospectral deformation of $J_{V/g}(x)$ and vice versa, hence, by the inverse problem, one of them can be obtained by a shift in x of another [17, 22, 38]).

We begin a direct proof of this assertion by recalling necessary results of spectral theory of finite band Jacobi matrices and related facts of complex analysis on Riemann surfaces (see e.g. [7, 17, 22, 38]).

Given the set σ of (2.1), denote Γ the two sheeted (hyperelliptic) Riemann surface, defined by the equation

$$w^2 = R(z),\ R(z) = \prod_{l=1}^{q}(z-a_l)(z-b_l),$$

i.e. obtained by pasting together two copies of the complex plane along the union of the "gaps" $(b_1, a_2), ..., (b_{q-1}, a_q), (b_q, a_1)$ of σ, the last gap goes through the infinity point. Let idp be the normalized differential of the third kind with simple poles of residues ± 1, at the infinity points $P_\pm$ on each sheet of Γ, and let $U = (U_1, ..., U_{q-1})$ be the vector of b-periods of dp:

$$U_l = \frac{1}{2\pi}\int_{b_l} dp,\ l = 1, ..., q-1, \tag{A.1}$$

where $\{b_l\}_{l=1}^{q-1}$ are the so-called b-cycles on Γ.

On the other hand, the integral

$$\int_{P_0}^{P} idp, \quad P_0, P \in \Gamma$$

with a properly chosen initial point P_0 can be identified with the complex Green function $G(z)$ of $\mathbb{C}\setminus\sigma$ with the pole at infinity (see e.g. [7]). The real part $g(z) = \Re G(z)$ is uniquely determined by the requirements to vanish for its limiting values on σ and to be harmonic in $\mathbb{C}\setminus\sigma$ for $g(z) - \log|z|$. It follows then that if ν is the unique minimizer of (2.2), hence solves the corresponding Euler-Lagrange equation (2.3), then

$$g(z) = \int_s \log|z-\mu|\nu(d\mu) - l_\sigma/2.$$

This and (A.1) imply (2.6), where $\alpha_l - \alpha_{l+1} = N([a_{l+1}, b_{l+1}]),\ l = 1, ..., q-1$ is the harmonic measure at infinity of the $(l+1)$th "band" $[a_{l+1}, b_{l+1}]$ of σ.

Denote $\theta : \mathbb{T}^{q-1} \to \mathbb{C}$ the Riemann θ-function, associated with Γ. Then according to [40, 7, 33] the leading coefficient γ_n of the polynomial p_n, where $\{p_l\}_{l\geq 0}$ are orthonormal polynomials on σ with respect to weighs, satisfying (2.7), is for $n \to \infty$:

$$\gamma_n^2 = A_\sigma e^{nl_\sigma}\left[\frac{\theta(n\alpha + u(\infty) + d_\sigma)}{\theta(n\alpha - u(\infty) + d_\sigma)} + o(1)\right]. \tag{A.2}$$

Here l_σ is defined in (2.3),

$$u(z)=\int_{b_q}^{z}\omega$$

with the integral taken along a path on the first sheet and $\omega=(\omega_1,...,\omega_{q-1})$ is the canonical basis of the differential of the first kind on Γ, A_σ and d_σ do not depend on n but depend on σ, the weight, and the points $\zeta_1,...,\zeta_{q-1}$ of Γ that are the poles of the corresponding Baker-Akhiezer function [17]. In the case, where $\alpha_l=m_l/q$ with positive integers $m_1,...,m_q$, hence with a q-periodic $J_\sigma(x)$ (see e.g. Section 4), $\zeta_1',...,\zeta_{q-1}'$ are the eigenvalues of the Dirichlet problem on the period for the corresponding finite-difference equation, distributed in a fixed way over the edges of the gaps. These are in fact the parameters, indexing representatives of the isospectral family. Another characterization of $\zeta_1',...,\zeta_{q-1}'$ is given in [7], Theorem W2.

Asymptotic formula (A.2) and the relation

$$r_n=\gamma_n/\gamma_{n+1}, \tag{A.3}$$

expressing the off-diagonal entries of a Jacobi matrix via the leading coefficients of associated orthonormal polynomials, lead to the relation

$$r_n^2=e^{-l_\sigma}\frac{\theta((n+1)\alpha-u(\infty)+d_\sigma)\theta(n\alpha+u(\infty)+d_\sigma)}{\theta((n+1)\alpha+u(\infty)+d_\sigma)\theta(n\alpha-u(\infty)+d_\sigma)}+o(1). \tag{A.4}$$

Replacing here n by $n+k$, where k is an arbitrary fixed integer (in fact $k=o(n)$), and passing to the limit (3.2), we obtain for the function $\mathcal{R}_\sigma$ of (3.4):

$$\mathcal{R}_\sigma(x)=e^{-l_\sigma}\frac{\theta(x+\alpha-u(\infty)+d_\sigma)\theta(x+u(\infty)+d_\sigma)}{\theta(x+\alpha+u(\infty)+d_\sigma)\theta(x-u(\infty)+d_\sigma)}. \tag{A.5}$$

By using the formula

$$\alpha+2u(\infty)=0, \tag{A.6}$$

that follows from the Riemann bilinear relations (see e.g [17], Section 6), we can write

$$\mathcal{R}_\sigma(x)=\mathcal{R}(x+x_\sigma), \tag{A.7}$$

where

$$\mathcal{R}(x)=e^{-l_\sigma}\frac{\theta(x+\alpha)\theta(x-\alpha)}{\theta^2(x)} \tag{A.8}$$

and

$$x_\sigma=-u(\infty)+d_\sigma. \tag{A.9}$$

Consider now the orthonormal polynomials $\{p_l^{(n)}\}_{l\geq 0}$ with respect to varying weights (1.10) – (1.11). Then we have for the leading coefficient of $p_n^{(n)}$ according to [15], formula (1.63):

$$\left(\gamma_n^{(n)}\right)^2=A_V e^{nl_V}\left[\frac{\theta(n\beta+u(\infty)+d_V)}{\theta(n\beta-u(\infty)+d_V)}+o(1)\right],\ n\to\infty, \tag{A.10}$$

where l_V is defined in (2.14) – (2.16), β is defined in (2.18), $u(\infty)$ is the same as in (A.2), and A_V and d_V do not depend on n but depend on V and the points $\zeta_1'',...,\zeta_{q-1}''$, that are zeros of a certain analytic function on $\mathbb{C}\setminus\sigma$ (see [15], formulas (1.26) – (1.27), (1.30)).

In view of the relations

$$r_{n+k}^{(n)} = \gamma_{n+k}^{(n)}/\gamma_{n+k+1}^{(n)} \tag{A.11}$$

(cf (A.3)) and (3.25) we need the coefficients $\gamma_{n+k}^{(n)},\ n \to \infty,\ k \in \mathbb{Z}$ fixed (see (3.22)) in order to find the entries of $J_V(x)$. We will find them by using the same trick as in obtaining (3.22). According to the trick the passage from n to $n+k,\ n \to \infty,\ k = o(n)$ can be carried out by passage from the super-index n to $n+k$, which is equivalent to the infinitesimal change $g \ \to \ g + gk/n$ in the inverse amplitude of the potential. Thus, replacing V by V/g in (A.10), using the above trick and (3.23) – (3.24), we obtain (cf (A.2)):

$$\left(\gamma_{n+k}^{(n)}\right)^2 = A_V e^{nl_V + k(gl_V)'} \left[\frac{\theta(n\beta + k\alpha + u(\infty) + d_V)}{\theta(n\beta + k\alpha - u(\infty) + d_V)} + o(1)\right],\ n \to \infty,$$

Now, comparing the Euler-Lagrange equations (2.14) – (2.16) for (2.12) and (2.3) – (2.3) for (2.2), we find the relation (cf (3.23) – (3.24)):

$$l_{\sigma_g} = (gl_{V/g})'.$$

This and (A.11) yield for the asymptotics of the off-diagonal entries of matrix $J^{(n)}$ (1.20), associated with $\{p_l^{(n)}\}_{l\geq 0}$:

$$\begin{aligned}\left(r_{n+k}^{(n)}\right)^2 &= \left(\gamma_{n+k}^{(n)}\right)^2 \left(\gamma_n^{(n)}\right)^{-2} \\ &= e^{-l_\sigma} \frac{\theta(n\beta + (k+1)\alpha - u(\infty) + d_V)\theta(n\beta + k\alpha + u(\infty) + d_V)}{\theta(n\beta + (k+1)\alpha + u(\infty) + d_V)\theta(n\beta + k\alpha - u(\infty) + d_V)} + o(1).\end{aligned}$$

Passing here to the limit (3.21), we obtain for the function $\mathcal{R}_V$ of (3.25), determining the off-diagonal entries of the limiting matrix $J_V(x)$:

$$\mathcal{R}_V(k\alpha + x) := \mathcal{R}_V(1, l\alpha(1) + x) = \mathcal{R}(k\alpha + x + x_V),$$

i.e.,

$$\mathcal{R}_V(x) = \mathcal{R}(x + x_V) \tag{A.12}$$

where $\mathcal{R}$ is defined in (A.8), and (cf (A.9))

$$x_V = -u(\infty) + d_{V/g}. \tag{A.13}$$

Comparing (A.7) and (A.12) we conclude that $\mathcal{R}_\sigma$ and $\mathcal{R}_V$ differ by a shift of argument.

The assertion, formulated at the beginning of the Appendix is proved.

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